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Random walks with power-law fluctuations in the number of steps

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Abstract

In this paper we derive limiting distributions for the resultant of an anisotropic random walk, where the individual steps comprising the walk are drawn from distributions of arbitrary but finite variance, and where the number of steps fluctuate according to a discrete power-law distribution. The consideration of discrete distributions with power-law tails is motivated by their recently discovered relevance to complex systems and networks. When this random walk is unbiased, a power-law decay for large values of the amplitude of the resultant occurs. For a small directional bias, the power-law tail persists only in the direction parallel to the bias, exhibiting an exponential decay in all other directions. We consider the relevance of the limiting distributions to non-diffusive transport.

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1. Introduction

The random walk is concept of considerable utility and impact in the quantitative sciences. The familiar central limit theorem of classical statistics [1] leads to Gaussian distributed variables when forming sums of arbitrarily distributed random variates of finite variance. In this sense, the Gaussian forms a limit distribution with a large basin of attraction. The Gaussian is a special case of the stable or Lévy [2] distributions that, by direct analogy, are the attractors of those distributions with infinite variance. Such distributions have been extensively used in recent years as models for critical phenomena [3] and complex systems [4], and are natural descriptors of scale-free random behaviour. Many situations require the consideration of random walks where the number of steps in the walk is itself a fluctuating variable. In this case, attraction to the Gaussian basin according to the tenets of the central limit theorem is not necessarily obtained. For example, a random walk with negative binomial step number fluctuations leads to the class of K-distributions [5], that has had considerable success in describing clutter in coherent imaging systems, among others. Likewise, when considering a random walk of stable distributed random variables with fluctuating step number, the generalization of the central limit theorem [6], does not necessarily lead to a stable distributed resultant. As demonstrated in [7], a random walk with stable distributed step lengths having negative binomial fluctuations in the number of steps obtains a class of infinitely divisible [1] distribution that can have two power-law regimes: one at small scales that smoothly changes to another at large scales.

Sand-pile cellular automata [8] provide a particularly simple algorithmic means for generating random behaviours that can be described by stable distributions. A particular rice-pile simulation [9] enables individual grains to be traced. The distance that a grain moves in an avalanche is approximately described by a stable distribution. This is, at first consideration, a paradox, for it implies that a grain can experience a 'flight' of arbitrary length which is apparently at variance with energetic constraints. Closer inspection reveals that the Lévy flight of the particle comprises a number of sub-flights whose length is a random variable with finite variance, but where the number of sub-flights fluctuate according to a distribution with a power-law tail. Thus, the power-law tail of the flight-length distribution is inherited from the number fluctuations rather than being a consequence of a classical Lévy flight. This prompts the questions of how the dimensionality of the space in which the walk occurs affects the form of the limiting distribution, and also what impact a bias in the walk has (grains move in a downward direction). These are the issues that underpin the calculations performed in this paper. The discrete power-law distributions are themselves a new discovery [9, 10], that have a relevance to the study of complex networks such as the World Wide Web (WWW) in addition to complex systems such as sand piles.

The analysis follows in the spirit of that adopted in [11], where a random walk was considered in n dimensions having an arbitrary directional bias with negative binomial number fluctuations, thereby obtaining a generalization of the K-distribution. Here, the number fluctuations have a power-law tail, the moments for which do not exist; a fact that necessarily leads to a different technical treatment to that adopted in [11], and to substantially different results.

This paper is organized as follows. In section 2 we consider an isotropic random walk in two dimensions, for clarity, and then we extend the study to more dimensions. In section 3 we do the same, introducing an anisotropy.

2. The isotropic random walk in two or more dimensions

The key tool for developing our theory is the characteristic function (CF), which is the Fourier transform of the probability density function (PDF) $p(\mathbf{R})$ of a random variable $\mathbf{R}$ that is a vector with n components, where n is the dimension of the space, i.e.

$$c(\mathbf{u}) = \int_{\mathbf{R}} p(\mathbf{R}) \exp(\mathrm{i}\mathbf{u}\cdot\mathbf{R})\,\mathrm{d}\mathbf{R} \tag{1}$$

$$\equiv \langle \exp(\mathrm{i}\mathbf{u}\cdot\mathbf{R})\rangle. \tag{2}$$

The angled brackets denote an ensemble average. If $c(\mathbf{u})$ is known, $p(\mathbf{R})$ can be obtained by Fourier inversion, namely

$$p(\mathbf{R}) = \frac{1}{(2\pi)^n} \int_{\mathbf{u}} c(\mathbf{u}) \exp(-\mathrm{i}\mathbf{u}\cdot\mathbf{R})\,\mathrm{d}\mathbf{u}. \tag{3}$$

We consider random walks where the individual steps are statistically identical, but the number of steps comprising the walk fluctuate with each realization. The characteristic function for such a walk with N steps is $[c(\mathbf{u})]^N$ for each realization of N. Averaging this expression over the distribution for N obtains an average characteristic function, which we denote as $\overline{c_N(\mathbf{u})}$. The distribution of N with which we are exclusively concerned is the discrete power law

$$p(N) = \frac{1}{\zeta(\beta)N^{\beta}} \tag{4}$$

where $\zeta(\beta)$ is the Riemann Zeta function [12],

$$\zeta(\beta) = \sum_{k=1}^{\infty} k^{-\beta} \tag{5}$$

and provides the normalization, $\beta > 1$ is a real number and $N \geqslant 1$. This choice comes from the behaviour of a sand-pile model [9], where N is the number of sub-flights that comprise a total avalanche flight performed by one grain.

The average value of N is

$$\bar{N} \equiv \langle N \rangle = \frac{1}{\zeta(\beta)} \sum_{N=1}^{\infty} \frac{1}{N^{\beta-1}} = \frac{\zeta(\beta-1)}{\zeta(\beta)} \tag{6}$$

and $\bar{N}$ exists provided that $\beta > 2$.

2.1. Isotropic random walk in two dimensions

In two dimensions, we consider the random walk to take place in the complex plane, so that the resultant vector $\mathbf{R} = \sum_{j=1}^{N} \mathbf{r}_j$, where $\mathbf{r}_j$ is the jth step, can be written as the complex number

$$\mathbf{R} \equiv R\exp(\mathrm{i}\phi) = \sum_{j=1}^{N} r_j \exp(\mathrm{i}\varphi_j) \tag{7}$$

where the step lengths r_j are statistically similar and independent of each other, i.e. $\langle r_j r_k \rangle = \langle r^2 \rangle \delta_{jk}$, where δ_{jk} is the Kronecker symbol. The mean value of r is

$$\langle r \rangle = \int_0^{\infty} p(r) r \, \mathrm{d}r. \tag{8}$$

The phases φ_j are independent of r_j, independent of each other and uniformly distributed over 2π, so that

$$p(\varphi) = \frac{1}{2\pi} \qquad 0 \leqslant \varphi \leqslant 2\pi. \tag{9}$$

2.1.1. Expression of the PDF for **R**. The PDF for **R** is found by Fourier inversion of the characteristic function. This is best achieved by first writing $\mathbf{r}_j$, $\mathbf{u}$ and $\mathbf{R}$ as complex numbers, whereupon

$$\mathbf{R} \cdot \mathbf{u} = u \sum_{j=1}^{N} r_j \cos(\varphi_j + \chi) \tag{10}$$

and the CF of this N-step random walk is then

$$\begin{aligned} c_N(\mathbf{u}) &= \left\langle \exp\left[\mathrm{i}u \sum_{j=1}^{N} r_j \cos(\varphi_j + \chi) \right] \right\rangle \\ &= \prod_{j=1}^{N} \langle \exp[\mathrm{i}u r_j \cos(\varphi_j + \chi)] \rangle. \end{aligned} \tag{11}$$

Since each step is statistically similar and independent, it follows that

$$c_N(\mathbf{u}) = \langle \exp[\mathrm{i}ur\cos(\varphi+\chi)]\rangle^N. \tag{12}$$

The CF of the single step is given by

$$\begin{aligned} c(\mathbf{u}) &= \langle \exp[\mathrm{i}ur\cos(\varphi+\chi)]\rangle_{\varphi,r} \\ &= \int_0^{2\pi} \mathrm{d}\varphi\, p(\varphi) \int_0^\infty \mathrm{d}r\, p(r) \exp[\mathrm{i}ur\cos(\varphi+\chi)]. \end{aligned} \tag{13}$$

The average over the φ variable gives

$$\langle \exp[\mathrm{i}ur\cos(\varphi+\chi)]\rangle_\varphi = J_0(ur) \tag{14}$$

where J_0 is a Bessel function of the first kind [12, 13]. Thus, the CF of the single step is

$$c(\mathbf{u}) = \langle J_0(ur)\rangle_r \tag{15}$$

and it is finite because we have assumed that all the moments of the distribution of the step lengths exist.

The CF for N steps is given by

$$c_N(\mathbf{u}) = \langle J_0(ur)\rangle_r^N \tag{16}$$

so if we now let N fluctuate according to equation (4), the averaged CF is

$$\overline{c_N(\mathbf{u})} = \sum_{N=1}^{\infty} \frac{1}{\zeta(\beta)N^\beta} \langle J_0(ur)\rangle_r^N \tag{17}$$

or equivalently

$$\overline{c_N(\mathbf{u})} = \frac{\langle J_0(ur)\rangle_r}{\zeta(\beta)} \sum_{N=0}^{\infty} \frac{\langle J_0(ur)\rangle_r^N}{(N+1)^\beta}. \tag{18}$$

The sum in equation (18) can be written exactly using a Lerch function Ψ [13]

$$\overline{c_N(\mathbf{u})} = \frac{\langle J_0(ur)\rangle_r}{\zeta(\beta)} \Psi\left(\langle J_0(ur)\rangle_r, \beta, 1\right) \tag{19}$$

and Fourier inversion of this expression obtains the PDF of $\mathbf{R}$

$$p_{\bar{N}}(\mathbf{R}) = \frac{1}{2\pi\zeta(\beta)} \int_0^\infty u J_0(Ru) \langle J_0(ru)\rangle_r \Psi(\langle J_0(ru)\rangle_r, \beta, 1)\, \mathrm{d}u. \tag{20}$$

It is possible to rewrite the Lerch function using the expression

$$\Psi(\langle J_0(ru)\rangle_r, \beta, 1) = \frac{1}{\Gamma(\beta)} \int_0^\infty \frac{t^{\beta-1}\,\mathrm{d}t}{\mathrm{e}^t - \langle J_0(ru)\rangle_r} \tag{21}$$

where Γ is the Gamma function. Upon interchanging the order of integration, the PDF then becomes

$$p_{\bar{N}}(\mathbf{R}) = \frac{1}{2\pi\zeta(\beta)\Gamma(\beta)} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} t^{\beta-1} \int_0^\infty \frac{J_0(Ru)u\,\mathrm{d}u}{\langle J_0(ru)\rangle_r^{-1} - \mathrm{e}^{-t}}. \tag{22}$$

This is a general result for the PDF of the resultant of a two-dimensional isotropic random walk with power-law fluctuating step number and is valid provided the characteristic function satisfies the condition

$$\lim_{u\to\infty} \langle J_0(ru)\rangle \sim O(u^{-\alpha}) \tag{23}$$

where $\alpha > 0$.

2.1.2. Asymptotics of the PDF. Examining the expression (22) for small u provides the behaviour of the PDF for large R. Expanding the Bessel function for small u obtains [12]

$$\langle J_0(ru)\rangle_r \simeq 1 - \frac{\langle r^2\rangle}{4}u^2 + \mathcal{O}\left(\langle r^4\rangle u^4\right), \tag{24}$$

whereupon the integral over the variable u appearing in equation (22) can be evaluated

$$\begin{aligned}\mathcal{I}_2(t,\mathbf{R}) &= \int_0^\infty \frac{J_0(Ru)u\,\mathrm{d}u}{\langle J_0(ru)\rangle_r^{-1} - \mathrm{e}^{-t}} \\ &= \frac{4}{\langle r^2\rangle}K_0\left(\left[\frac{4R^2}{\langle r^2\rangle}(1-\mathrm{e}^{-t})\right]^{1/2}\right)\end{aligned} \tag{25}$$

where K_0 is a modified Bessel function of the second kind, whence

$$p_{\bar{N}}(\mathbf{R}) \simeq \frac{2}{\pi\zeta(\beta)\Gamma(\beta)\langle r^2\rangle}\int_0^\infty \mathrm{e}^{-t}t^{\beta-1}K_0\left(\left[\frac{4R^2}{\langle r^2\rangle}(1-\mathrm{e}^{-t})\right]^{1/2}\right)\mathrm{d}t. \tag{26}$$

Setting $s^2 = 4R^2(1-\mathrm{e}^{-t})/\langle r^2\rangle$ gives

$$\mathcal{F}(\mathbf{R}) = \int_0^{2R/\sqrt{\langle r^2\rangle}} \frac{\langle r^2\rangle}{2R^2}\left[-\ln\left(1-\frac{\langle r^2\rangle}{4R^2}s^2\right)\right]^{\beta-1} K_0(s)s\,\mathrm{d}s \tag{27}$$

and letting $R\to\infty$ enables the logarithm to be expanded

$$\begin{aligned}\mathcal{F}(\mathbf{R}) &= \frac{\langle r^2\rangle}{2R^2}\int_0^\infty\left(\frac{\langle r^2\rangle}{4R^2}s^2\right)^{\beta-1} K_0(s)s\,\mathrm{d}s \\ &= \frac{\langle r^2\rangle^\beta[\Gamma(\beta)]^2}{2}\frac{1}{R^{2\beta}}\end{aligned} \tag{28}$$

using [13]. The tail of the PDF therefore behaves as

$$p_{\bar{N}}(\mathbf{R}) \simeq \frac{\Gamma(\beta)\langle r^2\rangle^{\beta-1}}{\pi\zeta(\beta)}\frac{1}{R^{2\beta}} \qquad R\to\infty. \tag{29}$$

We can see from this result that the PDF of the resultant of a two-dimensional random walk where the individual step lengths have all finite moments can nevertheless have a power-law tail, if the step number fluctuates with a power law. The index of the power-law tail is inherited from the number fluctuations.

The tail of the distribution of the modulus of $\mathbf{R}$ is obtained by integrating over the phase ϕ

$$p_{\bar{N}}(R) = \int_0^{2\pi} p_{\bar{N}}(\mathbf{R})R\,\mathrm{d}\phi \tag{30}$$

and gives

$$p_{\bar{N}}(R) \simeq \frac{2\Gamma(\beta)\langle r^2\rangle^{\beta-1}}{\zeta(\beta)}\frac{1}{R^{2\beta-1}} \qquad R\to\infty. \tag{31}$$

To evaluate the behaviour of the distribution for $R \ll 1$, we expand the factor $J_0(Ru)$ appearing in equation (22)

$$J_0(Ru) \simeq 1 - \frac{R^2u^2}{4} \tag{32}$$

whereupon to leading order in $\mathbf{R}$

$$p_{\bar{N}}(\mathbf{R}) = \frac{1}{2\pi\zeta(\beta)\Gamma(\beta)}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t}t^{\beta-1}\int_0^\infty \frac{u\,\mathrm{d}u}{\langle J_0(ru)\rangle_r^{-1} - \mathrm{e}^{-t}} \tag{33}$$

which does not depend on R, so

$$p_{\bar{N}}(\mathbf{R}) \simeq C_2 \qquad R \to 0 \tag{34}$$

where C_2 is a constant. From equation (30), it follows that

$$p_{\bar{N}}(R) \sim R \qquad R \to 0. \tag{35}$$

To summarize, the asymptotic behaviour of the PDF of the resultant of an isotropic two-dimensional random walk where the step lengths have a regular PDF and the step number fluctuates as a power law has the following properties:

$$p_{\bar{N}}(\mathbf{R}) \sim R^{-2\beta} \qquad R \to \infty \tag{36}$$

$$p_{\bar{N}}(\mathbf{R}) \sim 1 \qquad R \to 0 \tag{37}$$

$$p_{\bar{N}}(R) \sim R^{1-2\beta} \qquad R \to \infty \tag{38}$$

$$p_{\bar{N}}(R) \sim R \qquad R \to 0. \tag{39}$$

2.2. *Isotropic random walk in n dimensions*

The derivation of section 2.1 can be generalized to n dimensions, and we report here the principal results. The CF of the single step is given by

$$c(\mathbf{u}) = \left(\frac{n}{2} - 1\right)! \left(\frac{2}{u}\right)^{\frac{n}{2}-1} \int_0^\infty \frac{J_{\frac{n}{2}-1}(ur)}{r^{\frac{n}{2}-1}} p(r)\,\mathrm{d}r \tag{40}$$

where we have used

$$\int_{\Omega_n} \mathrm{d}\Phi_n \exp(\mathrm{i}\mathbf{u}\cdot\mathbf{r}) = \frac{(2\pi)^{\frac{n}{2}}}{(ur)^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(ur). \tag{41}$$

The PDF of the resultant $\mathbf{R}$ of an isotropic n-dimensional random walk with regular PDF for the step lengths and power-law PDF for the step number is then

$$p_{\bar{N}}(\mathbf{R}) = \frac{1}{(2\pi)^{\frac{n}{2}} R^{\frac{n}{2}-1} \zeta(\beta)\Gamma(\beta)} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} t^{\beta-1} \int_0^\infty \frac{u^{\frac{n}{2}} J_{\frac{n}{2}-1}(Ru)\,\mathrm{d}u}{[c(\mathbf{u})]^{-1} - \mathrm{e}^{-t}} \tag{42}$$

which is valid for $n \geqslant 2$ provided the Hankel transform of the function $u^{(n-1)/2}[c(\mathbf{u})]$ [14] exists. The one-dimensional random walk is qualitatively different because the random walker can only move back and forth. This has been considered in [9].

The asymptotics of equation (42) can be determined:

$$p_{\bar{N}}(\mathbf{R}) \sim R^{-n-2\beta+2} \qquad R \to \infty \tag{43}$$

$$p_{\bar{N}}(\mathbf{R}) \sim 1 \qquad R \to 0 \tag{44}$$

$$p_{\bar{N}}(R) \sim R^{1-2\beta} \qquad R \to \infty \tag{45}$$

$$p_{\bar{N}}(R) \sim R^{n-1} \qquad R \to 0. \tag{46}$$

Note that for the vector $\mathbf{R}$ the power-law decay depends on the dimension of the space $n \geqslant 2$ and the index characterizing the number fluctuations β, whereas the magnitude of $\mathbf{R}$ depends on β alone.

3. Random walk with directional bias

In this section, we determine the PDF of the resultant $\mathbf{R}$ of a random walk where the step lengths r have a regular PDF, the step numbers have a power-law distribution (equation (4)) and the steps and variables are all independent, as in the previous section. But this time the PDF of the phase $\hat{\mathbf{r}}$ is not uniform: $p(\hat{\mathbf{r}}) \equiv f(\hat{\mathbf{r}})$, where $f(\hat{\mathbf{r}})$ measures a directional bias for the steps comprising the random walk, as, for example, a confining magnetic field would do for the particles in a plasma. We assume that f can be decomposed into two terms

$$f(\hat{\mathbf{r}}) = f_0 + f_1(\hat{\mathbf{r}}) \tag{47}$$

where f_0 is independent of the phase. We perform a first-order expansion in f_1/f_0, which is equivalent to a perturbation analysis about the isotropic case.

3.1. Biased random walk in two dimensions

The details of the calculation are more intricate than before, therefore we provide details for the more intelligible random walk in two dimensions and we quote the results for the n-dimensional case.

3.1.1. Expression for the PDF of $\mathbf{R}$. In two dimensions, assuming the same variables as in section 2.1, we can write $f(\hat{\mathbf{r}}) = f(\varphi)$ and, since f is a PDF, we can write the identity

$$1 = \int_0^{2\pi} f(\varphi)\,\mathrm{d}\varphi \tag{48}$$

$$= 2\pi f_0 + \int_0^{2\pi} f_1(\varphi)\,\mathrm{d}\varphi \tag{49}$$

so that

$$f(\varphi) = \frac{1 + f_1(\varphi)}{2\pi\left[1 + \frac{1}{2\pi}\int_0^{2\pi} f_1(\varphi)\,\mathrm{d}\varphi\right]} \tag{50}$$

on transforming $f_1 \to f_1/f_0$. If we assume that $\beta > 2$, then $\bar{N}$ exists, so we can rescale again $f_1 \to f_1/\bar{N}^{\frac{1}{2}}$, and expand the right-hand side of equation (50) for $\bar{N} \to \infty$, to give

$$f(\varphi) \simeq \frac{1}{2\pi}\left[1 + \frac{f_1(\varphi)}{\bar{N}^{\frac{1}{2}}} - \frac{1}{2\pi\bar{N}^{\frac{1}{2}}}\int_0^{2\pi} f_1(\varphi)\,\mathrm{d}\varphi\right]. \tag{51}$$

The CF of a single step in such a random walk is

$$c(\mathbf{u}) = \langle \exp(\mathrm{i}\mathbf{r}\cdot\mathbf{u})\rangle \tag{52}$$

$$= \int_0^\infty p(r)\,\mathrm{d}r \int_0^{2\pi} \exp(\mathrm{i}\mathbf{r}\cdot\mathbf{u}) f(\varphi)\,\mathrm{d}\varphi \tag{53}$$

and, using equation (51), the integral over φ gives

$$\begin{aligned}\langle \exp(\mathrm{i}\mathbf{r}\cdot\mathbf{u})\rangle_\varphi &= \int_0^{2\pi} \exp(\mathrm{i}\mathbf{r}\cdot\mathbf{u}) f(\varphi)\,\mathrm{d}\varphi \\ &= J_0(ur)\left[1 - \frac{1}{2\pi\bar{N}^{\frac{1}{2}}}\int_0^{2\pi} f_1(\varphi)\,\mathrm{d}\varphi\right] \\ &\quad + \frac{1}{2\pi\bar{N}^{\frac{1}{2}}}\int_0^{2\pi}\exp(\mathrm{i}\mathbf{r}\cdot\mathbf{u}) f_1(\varphi)\,\mathrm{d}\varphi.\end{aligned} \tag{54}$$

We can now rescale $\mathbf{r} \to \mathbf{r}/\bar{N}^{\frac{1}{2}}$ and, assuming $\bar{N}$ as large, expand the exponential

$$\exp\left(\frac{\mathrm{i}\mathbf{r}\cdot\mathbf{u}}{\bar{N}^{\frac{1}{2}}}\right) \simeq 1 + \mathrm{i}\frac{\mathbf{r}\cdot\mathbf{u}}{\bar{N}^{\frac{1}{2}}} - \frac{(\mathbf{r}\cdot\mathbf{u})^2}{2\bar{N}} \tag{55}$$

so that

$$\left\langle \exp\left(\frac{\mathrm{i}\mathbf{r}\cdot\mathbf{u}}{\bar{N}^{\frac{1}{2}}}\right)\right\rangle_{\varphi} \simeq J_0\left(\frac{ur}{\bar{N}^{\frac{1}{2}}}\right)\left[1-\frac{1}{2\pi\bar{N}^{\frac{1}{2}}}\int_0^{2\pi} f_1(\varphi)\,\mathrm{d}\varphi\right]+\frac{1}{2\pi\bar{N}^{\frac{1}{2}}}\int_0^{2\pi} f_1(\varphi)\,\mathrm{d}\varphi + \frac{\mathrm{i}}{2\pi\bar{N}}\int_0^{2\pi}(\mathbf{r}\cdot\mathbf{u})f_1(\varphi)\,\mathrm{d}\varphi. \tag{56}$$

Expanding also the Bessel function for $\bar{N}\to\infty$, we obtain the CF to $\mathcal{O}(1/\bar{N})$

$$c(\mathbf{u}) \simeq \int_0^{\infty} p(r)\,\mathrm{d}r\left[1-\frac{r^2u^2}{4\bar{N}}+\frac{\mathrm{i}}{2\pi\bar{N}}\int_0^{2\pi}(\mathbf{r}\cdot\mathbf{u})f_1(\varphi)\,\mathrm{d}\varphi\right]. \tag{57}$$

We can rescale both f_1 and $\mathbf{r}$ with $\bar{N}^{\frac{1}{2}}$ and write the CF of the single step as

$$c(\mathbf{u}) \simeq 1-\tfrac{1}{4}u^2\langle r^2\rangle+\mathrm{i}(\mathbf{u}\cdot\boldsymbol{\delta}) \tag{58}$$

where

$$\boldsymbol{\delta}=\frac{1}{2\pi}\int_0^{\infty}p(r)\,\mathrm{d}r\int_0^{2\pi}\mathbf{r}f_1(\varphi)\,\mathrm{d}\varphi \tag{59}$$

which is a measure of the anisotropy of the random walk.

The derivation of the PDF is now the same as in section 2.1, and gives

$$p_{\bar{N}}(\mathbf{R})=\frac{1}{(2\pi)^2}\frac{1}{\zeta(\beta)\Gamma(\beta)}\int_0^{\infty}t^{\beta-1}\,\mathrm{e}^{-t}\,\mathrm{d}t\int_{\mathbf{u}}\frac{\exp(-\mathrm{i}\mathbf{R}\cdot\mathbf{u})\,\mathrm{d}\mathbf{u}}{[c(\mathbf{u})]^{-1}-\mathrm{e}^{-t}}. \tag{60}$$

The integral over the variable u will exist provided that it satisfies the conditions for the existence of a Hankel transform [14]. Substituting for $c(\mathbf{u})$, we obtain

$$p_{\bar{N}}(\mathbf{R})\simeq\frac{1}{(2\pi)^2}\frac{1}{\zeta(\beta)\Gamma(\beta)}\int_0^{\infty}t^{\beta-1}\,\mathrm{e}^{-t}\,\mathrm{d}t \times\int_{\mathbf{u}}\frac{\exp(-\mathrm{i}\mathbf{R}\cdot\mathbf{u})\,\mathrm{d}\mathbf{u}}{1+\frac{1}{4}u^2\langle r^2\rangle-\mathrm{i}(\mathbf{u}\cdot\boldsymbol{\delta})-\mathrm{e}^{-t}}. \tag{61}$$

This is the general expression for the PDF of the resultant of a random walk where the step lengths are drawn from a regular PDF, the step numbers are power-law distributed and there is a directional bias whose effect is encapsulated in the parameter $\boldsymbol{\delta}$. We now evaluate the tail of this PDF.

3.1.2. Asymptotics of the PDF. Consider the integral in $\mathbf{u}$, appearing in equation (61), which with a change of variable to $\mathbf{v}=\mathbf{u}-2\mathrm{i}\boldsymbol{\delta}/\langle r^2\rangle$ can be written as

$$\begin{aligned}\mathcal{B}(\mathbf{R},t)&=\exp\left(\frac{2}{\langle r^2\rangle}\mathbf{R}\cdot\boldsymbol{\delta}\right)\int_{\mathbf{v}}\frac{\exp(-\mathrm{i}\mathbf{R}\cdot\mathbf{v})\,\mathrm{d}\mathbf{v}}{1+\frac{\delta^2}{\langle r^2\rangle}+\frac{\langle r^2\rangle}{4}v^2-\mathrm{e}^{-t}}\\&=\frac{8\pi}{\langle r^2\rangle}\exp\left(\frac{2}{\langle r^2\rangle}\mathbf{R}\cdot\boldsymbol{\delta}\right)K_0\left(\left[\frac{4R^2}{\langle r^2\rangle}\left(1+\frac{\delta^2}{\langle r^2\rangle}-\mathrm{e}^{-t}\right)\right]^{1/2}\right).\end{aligned} \tag{62}$$

Apart from a factor deriving from the angular integration, this can be compared with equation (25).

We now have to evaluate the integral in t, which with substitution $s=(1+\delta^2/\langle r^2\rangle-\mathrm{e}^{-t})\langle r^2\rangle/\delta^2$ gives

$$\mathcal{D}(\mathbf{R})=\frac{\delta^2}{\langle r^2\rangle}\int_1^{1+\frac{\langle r^2\rangle}{\delta^2}}\left\{-\ln\left[1+\frac{\delta^2}{\langle r^2\rangle}(1-s)\right]\right\}^{\beta-1}K_0\left(\frac{2\delta R}{\langle r^2\rangle}\sqrt{s}\right)\mathrm{d}s. \tag{63}$$

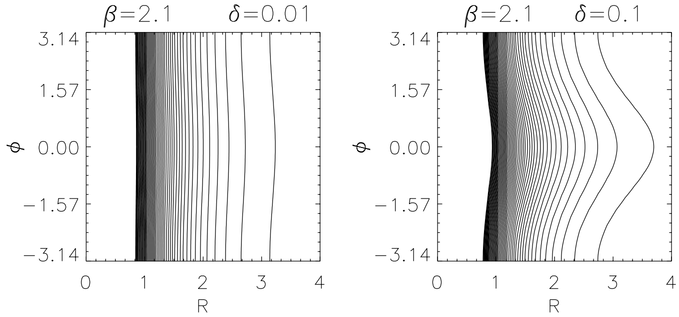

Figure 1. Contour plots of $p_{\bar{N}}(\mathbf{R})$, in the plane (R, ϕ), for $\beta = 2.1$ and different values of δ; $\langle r^2 \rangle = 1$ for all the graphics and there are 50 equally spaced equiprobability lines between 0 and 0.1 in each plot.

If we consider the limit $\delta^2/\langle r^2\rangle \ll 1$, we can expand the logarithm in $\delta^2/\langle r^2\rangle$ and replace the upper limit of integration by infinity:

$$\begin{aligned}\mathcal{D}(\mathbf{R}) &\simeq \left(\frac{\delta^2}{\langle r^2\rangle}\right)^{\beta} \int_1^{\infty} (1-s)^{\beta-1} K_0\left(\frac{2\delta R}{\langle r^2\rangle}\sqrt{s}\right) \mathrm{d}s \\ &= \Gamma(\beta)\left(\frac{\delta}{R}\right)^{\beta} K_\beta\left(\frac{2\delta R}{\langle r^2\rangle}\right). \end{aligned} \tag{64}$$

For large values of R, the expression of the PDF for $\mathbf{R}$ is

$$p_{\bar{N}}(\mathbf{R}) \simeq \frac{2}{\pi\zeta(\beta)\langle r^2\rangle}\left(\frac{\delta}{R}\right)^{\beta} K_\beta\left(\frac{2\delta R}{\langle r^2\rangle}\right)\exp\left(\frac{2}{\langle r^2\rangle}\mathbf{R}\cdot\boldsymbol{\delta}\right). \tag{65}$$

Notice that, if we let $\delta \to 0$, we obtain the isotropic result equation (29). We can expand the Bessel function for $R \to \infty$, and obtain

$$p_{\bar{N}}(\mathbf{R}) \simeq \frac{2}{\pi\zeta(\beta)\langle r^2\rangle}\left(\frac{\delta}{R}\right)^{\beta}\sqrt{\frac{\pi\langle r^2\rangle}{4\delta R}}\exp\left(-\frac{2R\delta}{\langle r^2\rangle}\right)\exp\left(\frac{2}{\langle r^2\rangle}\mathbf{R}\cdot\boldsymbol{\delta}\right). \tag{66}$$

So the tail of the distribution behaves as

$$p_{\bar{N}}(\mathbf{R}) \simeq \frac{\delta^{\beta-\frac{1}{2}}}{\sqrt{\pi\langle r^2\rangle}\zeta(\beta)}\frac{1}{R^{\beta+\frac{1}{2}}}\exp\left[-\frac{2\delta R}{\langle r^2\rangle}(1-\cos\theta)\right] \tag{67}$$

where θ is the angle between $\mathbf{R}$ and $\boldsymbol{\delta}$ and is equivalent to the mean direction in which the bias is directed. This PDF behaves as an exponential for large values of R for all θ, except for $\theta = 0$ where it has a power-law tail. Thus, for those directions where $\mathbf{R}$ is parallel to $\boldsymbol{\delta}$, the PDF has a power-law tail. For all other directions, the PDF has a multiplicative negative exponential dependence with a characteristic scale length $\langle r^2\rangle/(2\delta)$. Hence, moments of R will exist in all those directions other than those where $\mathbf{R}$ is parallel to $\boldsymbol{\delta}$.

Figure 1 shows the contour plots in the plane (R, ϕ) of $p_{\bar{N}}(\mathbf{R})$ for $\beta = 2.1$ and different values of δ, with $\langle r^2\rangle = 1$. These are calculated numerically using equation (61) and assuming, without loss of generality, that $\boldsymbol{\delta} \equiv \delta\hat{\boldsymbol{\phi}}$. We clearly see that when δ increases, keeping $\delta^2/\langle r^2\rangle \ll 1$, the contours become more distorted in the direction of $\hat{\phi}$ with increasing R.

The tail of the PDF for R can be evaluated

$$p_{\bar{N}}(R) \simeq \int_0^{2\pi} R\,\mathrm{d}\phi \frac{2}{\pi\zeta(\beta)\langle r^2\rangle}\left(\frac{\delta}{R}\right)^{\beta} K_\beta\left(\frac{2\delta R}{\langle r^2\rangle}\right)\exp\left(\frac{2}{\langle r^2\rangle}\mathbf{R}\cdot\boldsymbol{\delta}\right) \tag{68}$$

and, using

$$\begin{aligned}\int_0^{2\pi} \exp\left(\frac{2}{\langle r^2\rangle}\mathbf{R}\cdot\delta\right)\mathrm{d}\phi &= 2\pi J_0\left(\frac{2\mathrm{i}}{\langle r^2\rangle}R\delta\right)\\ &= 2\pi I_0\left(\frac{2}{\langle r^2\rangle}R\delta\right)\end{aligned} \tag{69}$$

where I_0 is a modified Bessel function of the first kind, we obtain

$$p_{\bar{N}}(R) \simeq \frac{4\delta^\beta}{\zeta(\beta)\langle r^2\rangle}\frac{1}{R^{\beta-1}}K_\beta\left(\frac{2\delta R}{\langle r^2\rangle}\right)I_0\left(\frac{2\delta R}{\langle r^2\rangle}\right). \tag{70}$$

Note that the result of equation (31) obtains as the biasing $\delta \to 0$.

3.2. Biased random walk in n dimensions

Finally we give the asymptotic results for the PDF of the resultant of an n-dimensional random walk where the step lengths have a regular PDF, the step numbers are power-law distributed and there is a directional bias.

When $R \to \infty$, the PDF of the resultant of an n-dimensional random walk with a directional bias is

$$p_{\bar{N}}(\mathbf{R}) \simeq \frac{2\delta_n^{\frac{n}{2}+\beta-1}}{\zeta(\beta)}\left(\frac{n}{2\pi\langle r^2\rangle}\right)^{\frac{n}{2}}\frac{\exp\left(\frac{n}{\langle r^2\rangle}\boldsymbol{\delta}_n\cdot\mathbf{R}\right)}{R^{\frac{n}{2}+\beta-1}}K_{\frac{n}{2}+\beta-1}\left(\frac{n\delta_n}{\langle r^2\rangle}R\right). \tag{71}$$

Expanding the Bessel function for $R \to \infty$, we obtain:

$$p_{\bar{N}}(\mathbf{R}) \simeq \frac{2\delta_n^{\frac{n}{2}+\beta-\frac{3}{2}}}{\zeta(\beta)}\left(\frac{2n}{\pi\langle r^2\rangle}\right)^{\frac{n}{2}-\frac{1}{2}}\frac{1}{R^{\frac{n}{2}+\beta-\frac{1}{2}}}\exp\left[-\frac{n\delta_n R}{\langle r^2\rangle}(1-\cos\theta)\right] \tag{72}$$

where $\boldsymbol{\delta}_n$ is the n-dimensional generalization of $\boldsymbol{\delta}$ defined in equation (59) and θ is the angle between $\boldsymbol{\delta}_n$ and $\mathbf{R}$. As in the two-dimensional biased case, the PDF has an exponential tail except for the direction parallel to the the bias, where it has a power-law decay.

4. Conclusions

In this paper, we have examined the effect of anisotropy or bias on random walks in an arbitrary dimension formed from step lengths of arbitrary distribution but finite variance, but where the number of steps fluctuate according to an inverse power law. The limiting distributions are manifestly not of the stable class, even though they possess the characteristic power-law tails in the direction approximately parallel to the bias. In contrast, in the direction orthogonal to the bias, the distribution has an exponential tail. Such behaviour is reminiscent of the diffusion of particles in a confining magnetic field, where motion parallel and perpendicular to the field direction exhibits very different behaviour and properties. Indeed, the transport of particles and energy throughout the complex field structures within a magnetic confinement device is known to be anomalous and can exhibit avalanching phenomenology (see [15] and references therein and [16, 17]), and the distributions derived here may have an impact in this regard. The distributions provide a single-fold description of the fluctuations, but do not describe how such fluctuations evolve.

The fundamental and independent question of how discrete power-law distributions can arise from a dynamical or stochastic process is under active consideration [18]. A solution of this problem would provide the tools required for studying the spatio-temporal evolution, for instance, of an initially coherent ensemble of particles subject to directionally dependent scale-free fluctuations.

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